Multidimensional scaling has benign landscape under mild rank relaxation

DACO, March 19, 2024

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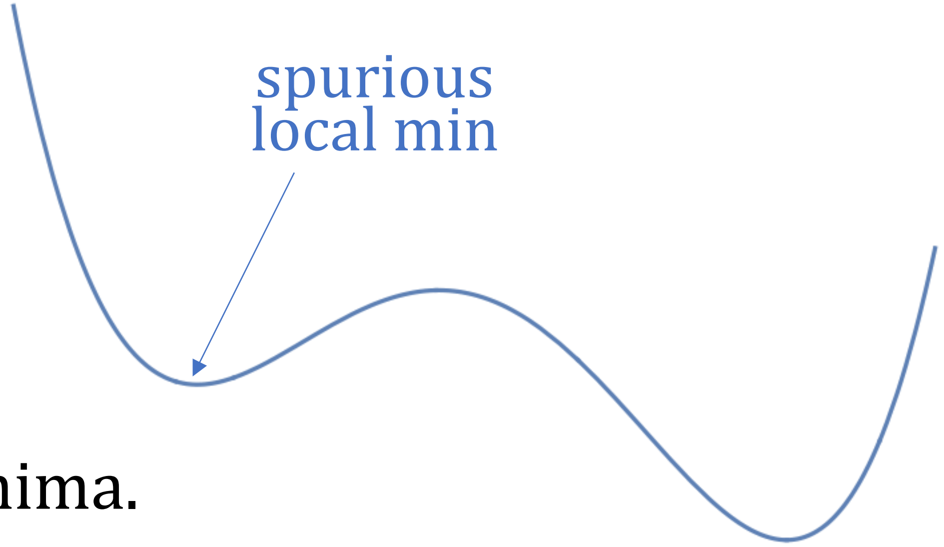

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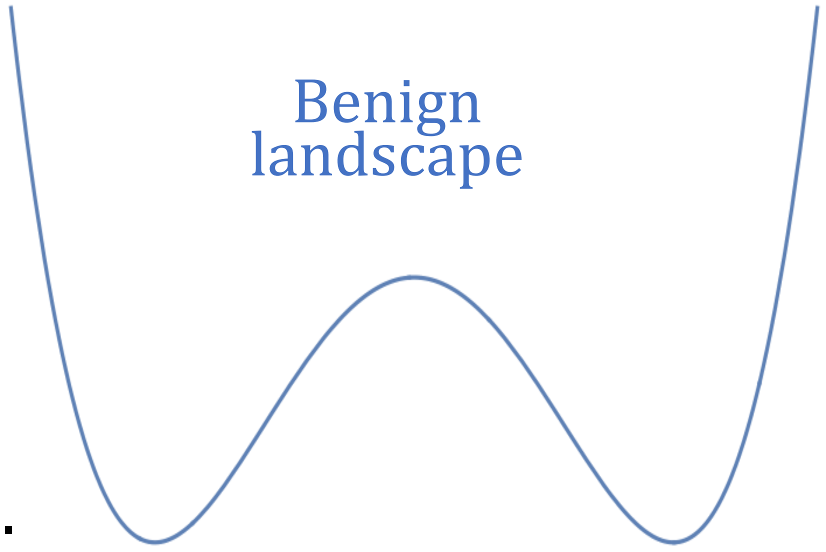

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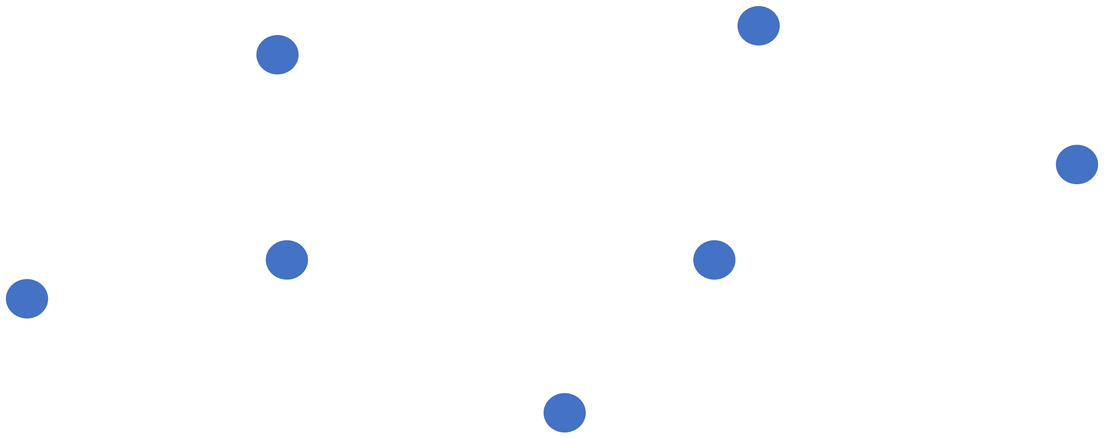
The problem

n unknown points $z_1^*, z_2^*, \dots, z_n^*$ in $\mathbb{R}^\ell$.



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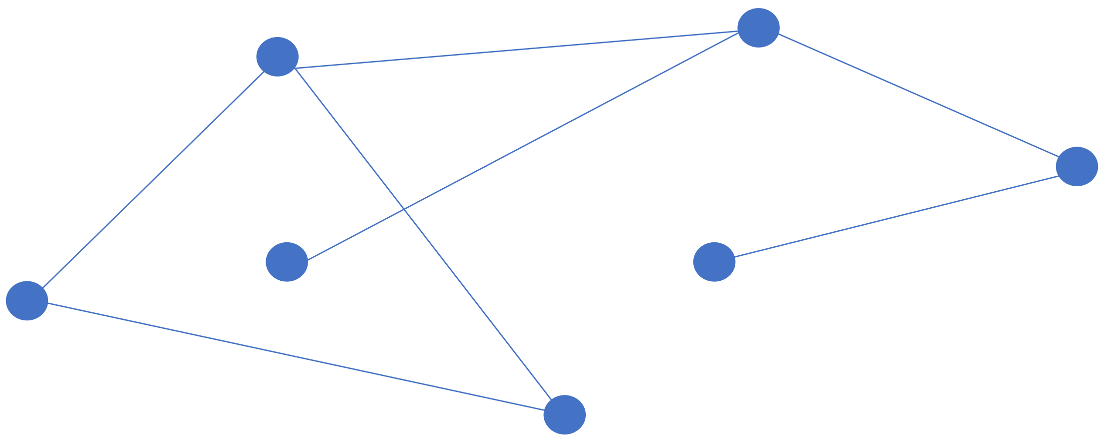

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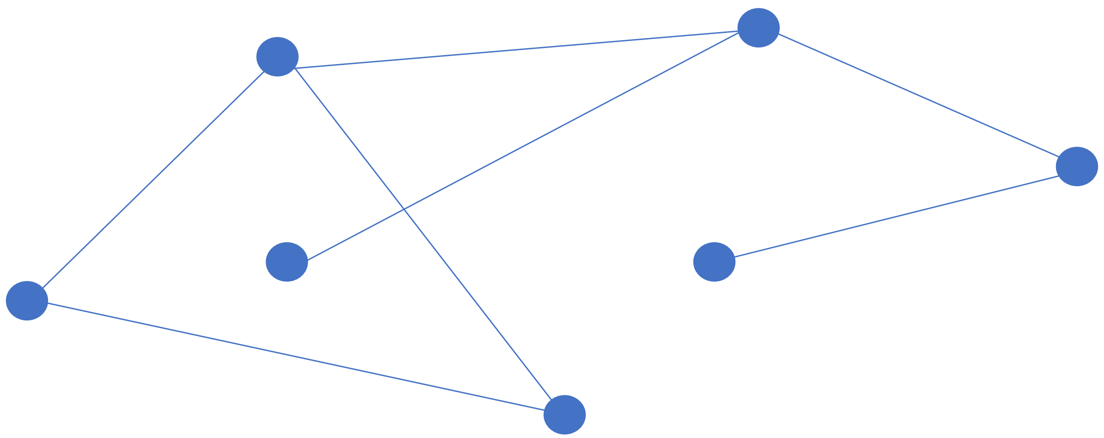
Goal: recover the n points (up to translation & rotation)



Applications

Robotics (**sensor network localization**), $\ell = 2,3$

Data analysis (metric **multidimensional scaling**)

Graph theory (rigidity)

Optimization problem

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"s-stress"

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Nonconvex! How bad?

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Possible variations: Noisy measurements, landmarks, …

Our focus: (nearly) complete graphs, no noise

Synthetic experiments

Recipe (all distances known):

(1) Choose ground truths $z_1^*, z_2^*, \ldots, z_n^*$ at random (normal iid)

(2) Run gradient descent/trust regions/etc.

(3) Find global min?

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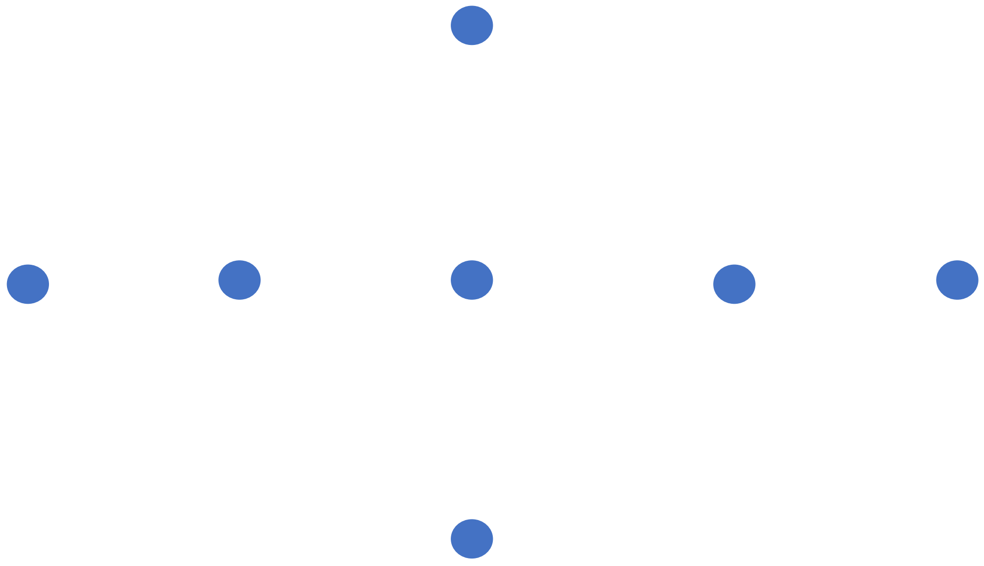
Open Question: Does s-stress have spurious local minima? Are all 2-critical points global minima?

* Malone & Trosset 2000, Parhizkar 2013, etc.

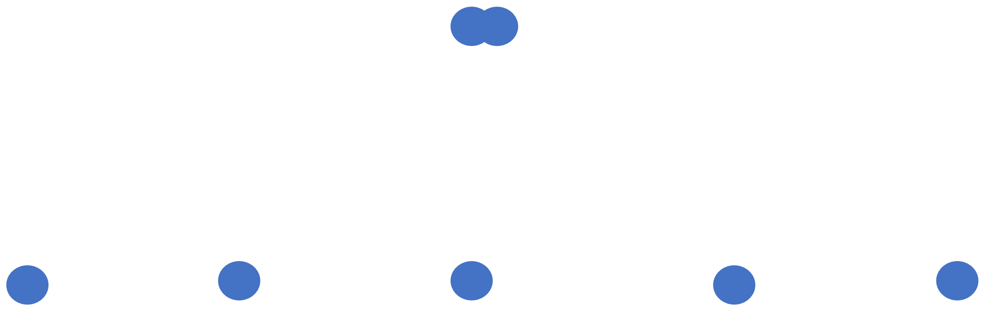
Counterexamples

s-stress can have spurious strict local minima!

Ground truth $z_1^*, z_2^*, \ldots$



Spurious configuration $z_1, z_1, \ldots$



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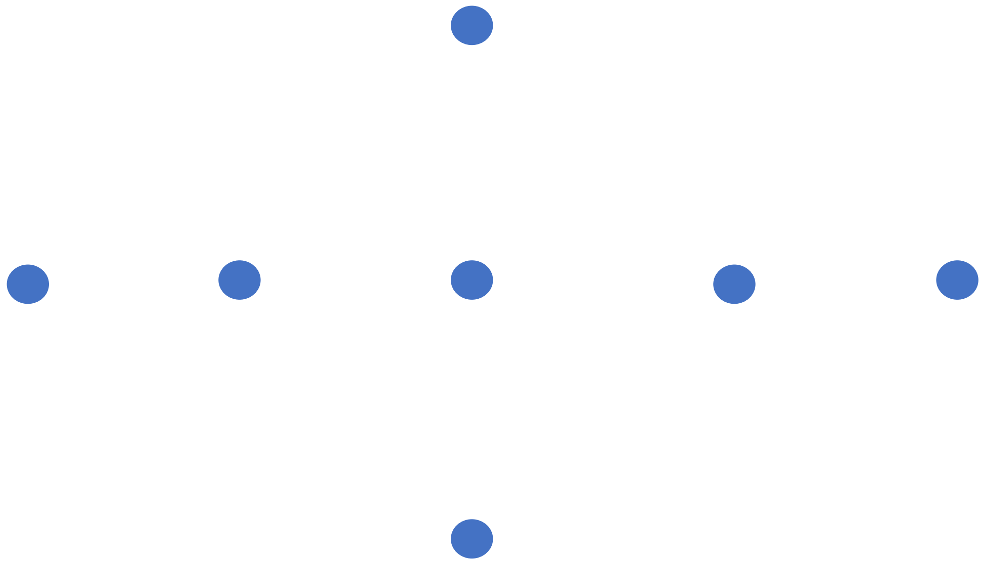

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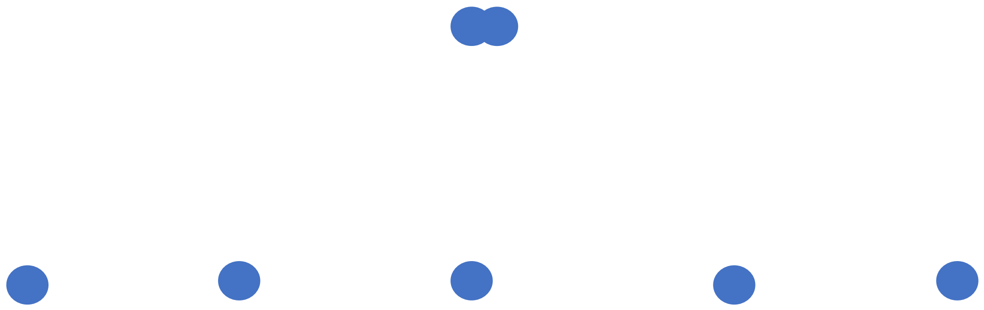

Set of ground truths with spurious local minima has positive measure

Nonconvex relaxation

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If $k = n - 1$, landscape is benign (later)
Can we do better?

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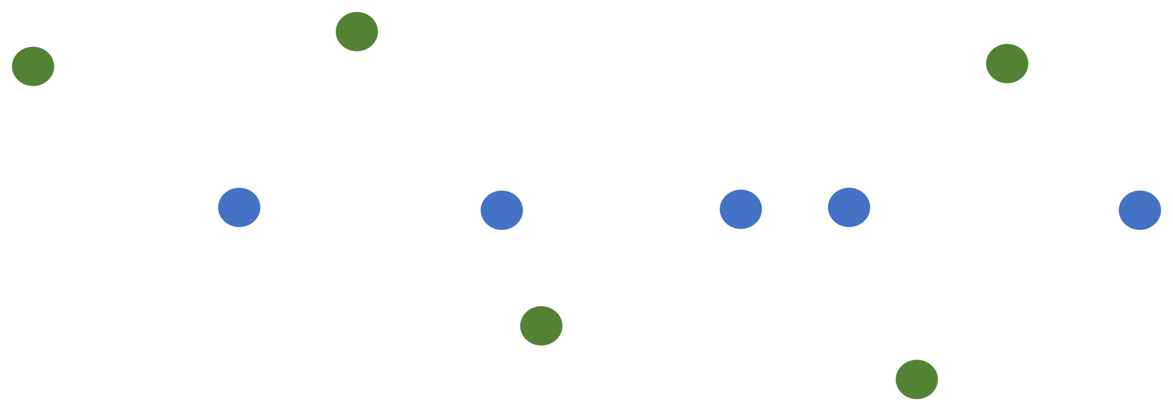

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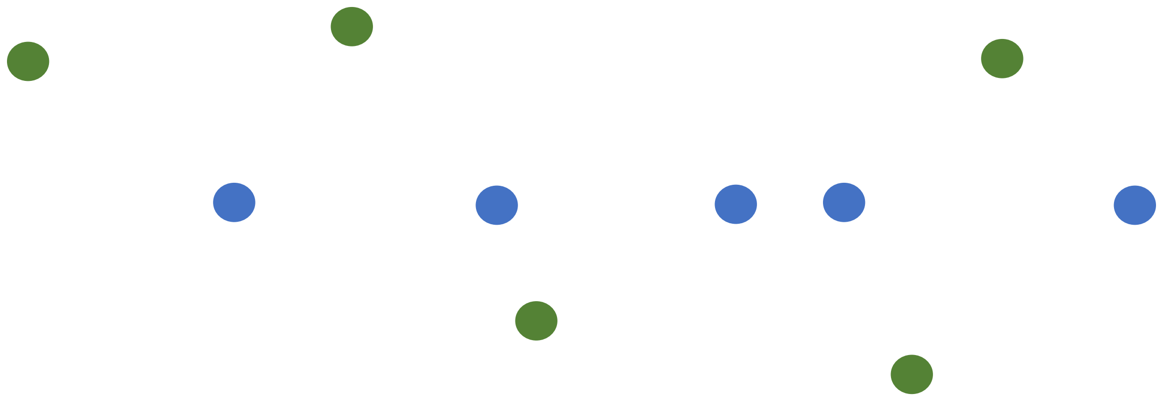

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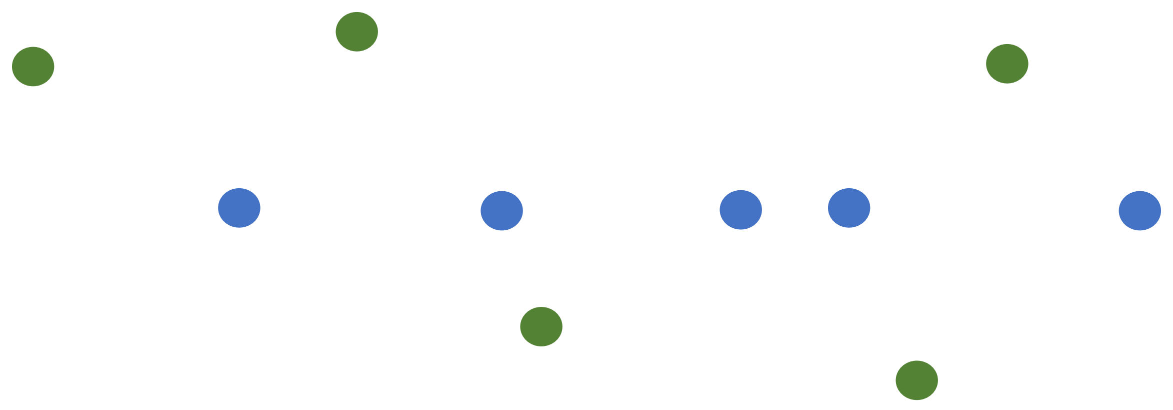

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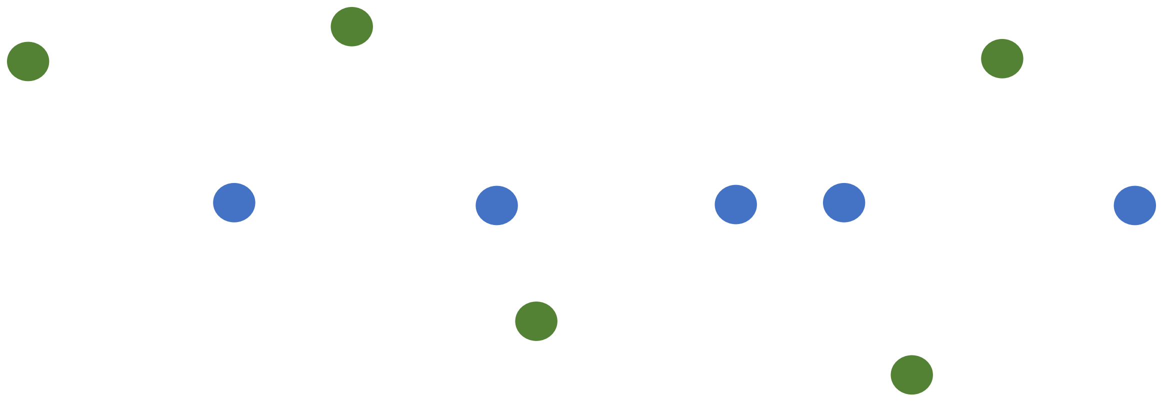

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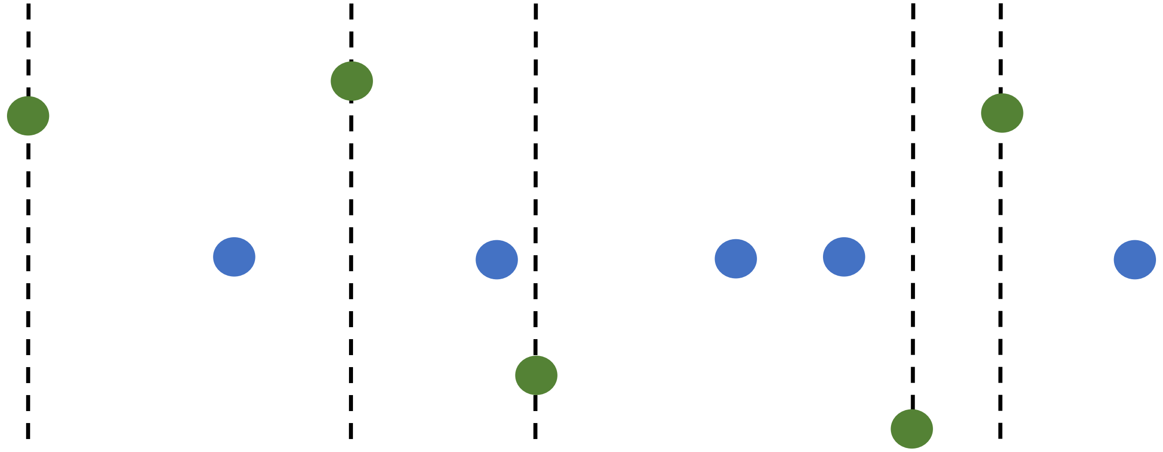

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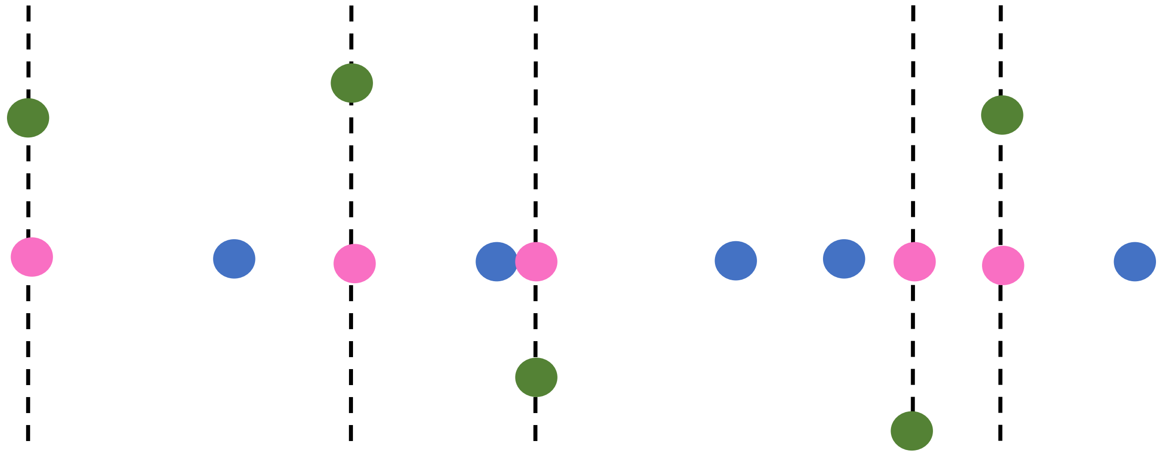

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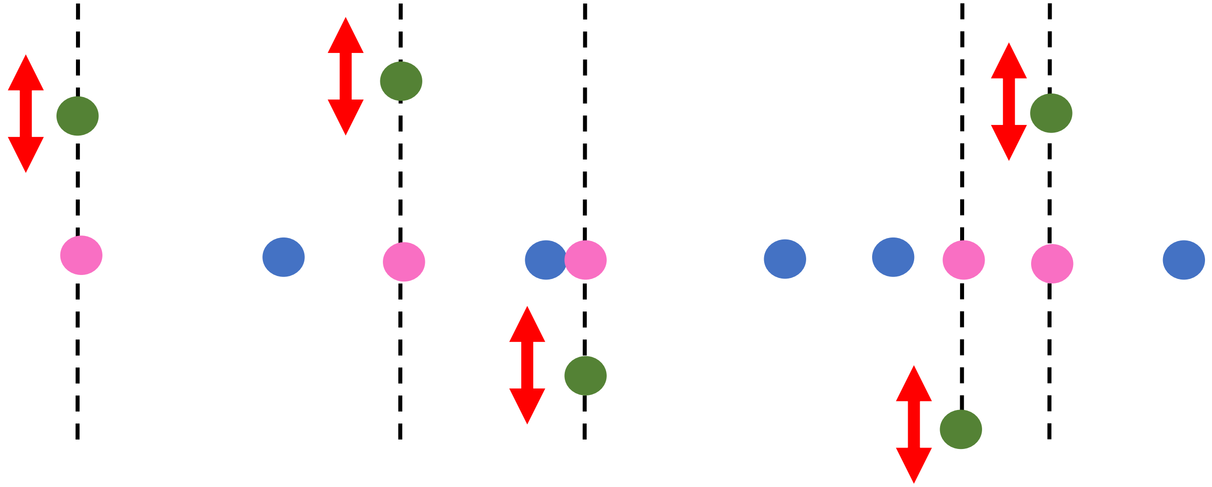

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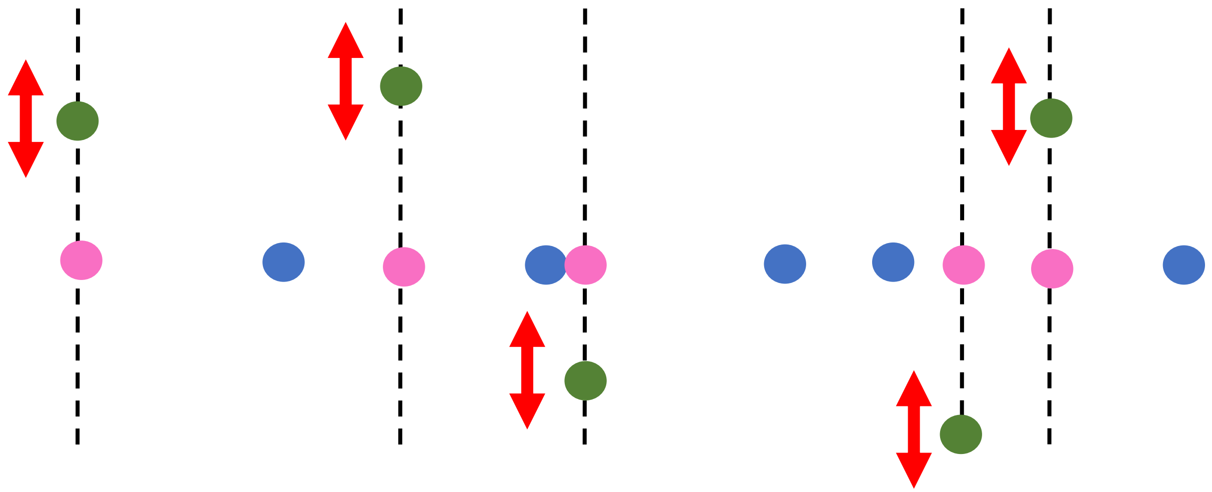

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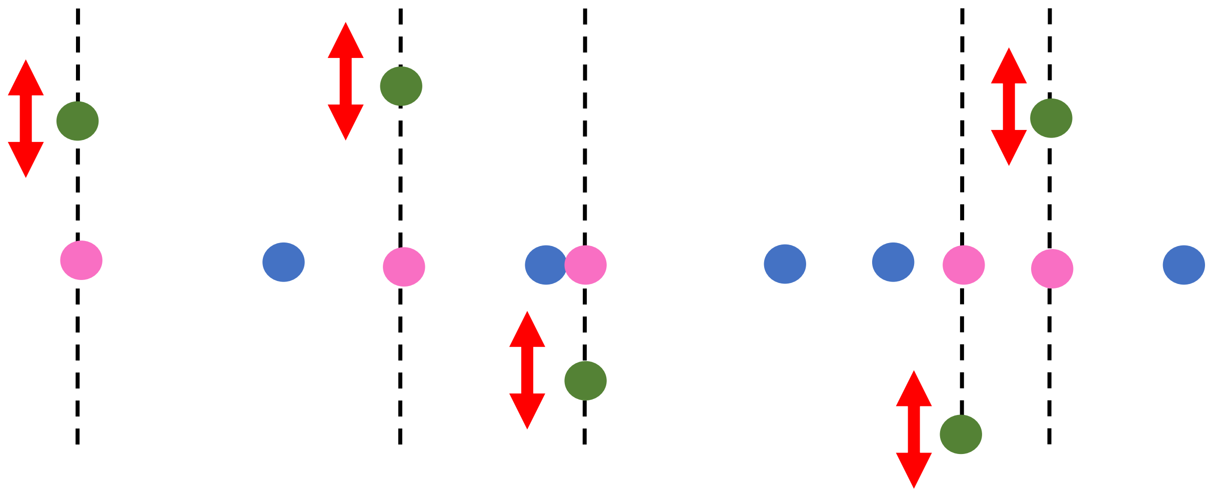

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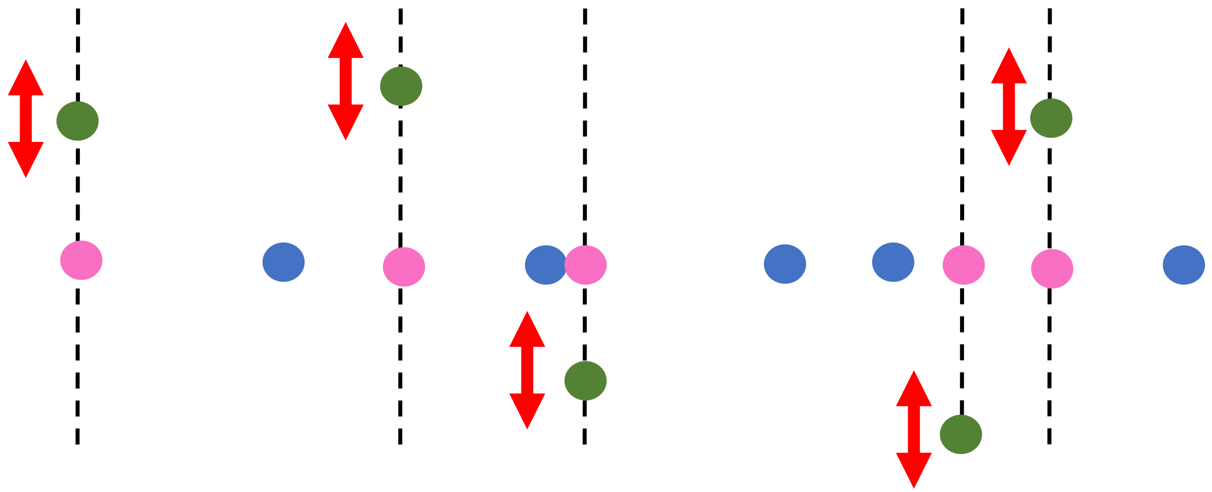

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For **isotropic GT,** $k \approx \ell \log(n)$, similar descent direction

Randomize over descent directions (instead of eigenvalue interlacing)

Connection to low-rank optimization

Notation and reformulation

$$Z = \begin{pmatrix} z_1^\top \\ \vdots \\ z_n^\top \end{pmatrix} \in \mathbb{R}^{n\times\ell}, \qquad Z_* = \begin{pmatrix} {z_1^*}^\top \\ \vdots \\ {z_n^*}^\top \end{pmatrix} \in \mathbb{R}^{n\times\ell}$$

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$$[\Delta(Y)]_{ij} := Y_{ii} + Y_{jj} - 2Y_{ij} = \left\| z_i - z_j \right\|^2$$

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Restricted Isometry Property? No! $\Delta^* \circ \Delta$ has condition number n

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General theorem: If Γ is completely positive, contractive, "behaves well w.r.t. rank-1 matrices", and

$$\langle Y, \Theta(Y) \rangle \leq c \langle Y, \Gamma(Y) \rangle \quad \forall Y$$

then landscape is benign when relax to $k \approx \ell + \sqrt{c\ell}$.

Open questions

Conjecture [**arbitrary GT**]: Relaxing to $k = \ell + 1$ is enough.

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Incomplete graphs (random, expanders, …)

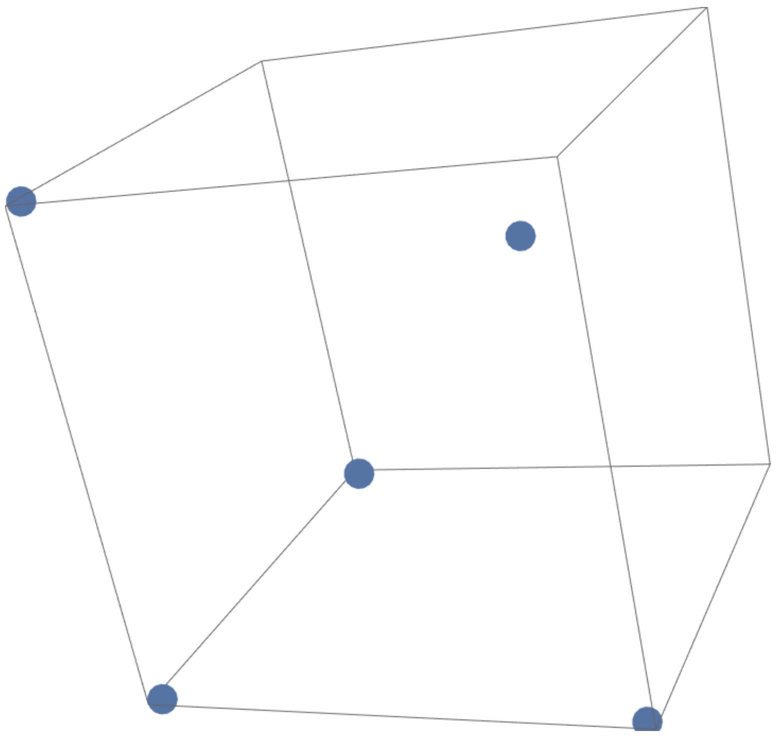
Many other SNL problems (noise models, trajectory localization, …)

Appendix

Counterexamples

Minima number of points to have spurious local minima?

$$\ell + 2 \text{ (for } \ell \geq 5\text{)}$$



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